The Journal of Computational Finance (27–47) Volume 12/Number 1, Fall 2008

Representing the CGMY and Meixner Lévy processes as time changed Brownian motions

Dilip B. Madan

Robert H. Smith School of Business, Van Munching Hall, University of Maryland, College Park, MD 20742, USA; email: dbm@rhsmith.umd.edu

Marc Yor

Laboratoire de probabilités et Modeles aléatoires, Université Pierre et Marie Curie, 4, Place Jussieu, F-75252 Paris Cedex, France; email: deaproba@proba.jussieu.fr

We describe the Carr–Geman–Madan–Yor (CGMY) and Meixner processes as time changed Brownian motions. The CGMY uses a time change that is absolutely continuous with respect to the one-sided stable $(Y/2)$ *subordinator while the Meixner time change is absolutely continuous with respect to the one-sided stable* $(1/2)$ *subordinator. The required time changes may be generated by simulating the requisite one-sided stable subordinator and throwing away some of the jumps as described by Asmussen and Rosinski (2001).*

1 INTRODUCTION

Lévy processes are increasingly being used to model the local motion of asset returns, permitting the use of distributions that are both skewed and capable of matching the high levels of kurtosis observed in factors driving equity returns. By way of examples we cite the normal inverse Gaussian process (Barndorff-Nielsen (1998)), the hyperbolic process (Eberlein *et al* (1998)), and the variance gamma process (Madan *et al* (1998)). For the valuation of structured equity products the importance of skewness is well recognized and has led to the development of local Lévy processes (see Carr *et al* (2004)) that preserve skews in forward implied volatility curves. It is also understood from the steepness of implied volatility curves that tail events have significantly higher prices than those implied by a Gaussian distribution with the consequence that pricing distributions display high levels of excess kurtosis.

On a single asset one may simulate the Lévy process calibrated to the prices of vanilla options to value equity structured products written on a single underlier. Such a simulation (see Asmussen and Rosinski (2001)) may approximate the small jumps using a diffusion process, with the large jumps simulated as a compound Poisson process where one uses the normalized large jump Lévy measure as the density of jump magnitudes with the integral of the Lévy measure over the large jumps serving as the jump arrival rate. However, increasingly one sees multiasset structures being traded and this requires asset correlations to be modeled.

Given marginal Lévy processes one can accommodate correlations if one can represent the Lévy process as a time changed Brownian motion. In this case we correlate the simulated processes by correlating the Brownian motions while preserving the independent time changes for each of the marginal underliers. For alternative approaches to introducing dependence in a multiasset Lévy context we refer the reader to Luciano and Schoutens (2006) and Joshi and Stacey (2005). For yet another approach to modeling dependence via correlating Brownian motions subjected to a common time change we refer the reader to the thesis of Chen (2008). An extensive study of the various ways to use these procedures in dependence studies is beyond the scope of this paper and is left to future research.

It is therefore useful to have representations of Lévy processes as time changed Brownian motions. For some Lévy processes, such as the variance gamma process or the normal inverse Gaussian process, these are known by construction of the Lévy process as a time changed Brownian motion. For other Lévy processes, such as the Carr–Geman–Madan–Yor (CGMY) process (Carr *et al* (2002); see also Koponen (1995); Boyarchenko and Levendorskii (1999, 2000)) or the Meixner process (Schoutens and Teugels (1998); see also Grigelionis (1999); Schoutens (2000) and Pitman and Yor (2003)), the process is defined directly by its Lévy measure and it is not clear *a priori* whether the processes can be represented as time changed Brownian motions. With a view to enhancing the applicability of these processes, particularly with respect to multiasset structured products, we develop the representations of these processes as time changed Brownian motions.

An additional advantage of the new representations is that of reducing processes of infinite variation, for example $Y > 1$ in CGMY, that are relatively harder to simulate (as explained by Asmussen and Rosinski (2001)) to time changed Brownian motion where the time change is a finite variation subordinator. The subordinator is then easier to simulate and the Brownian motion is also easily simulated.

Section 2 presents, for completeness, some preliminary results on Lévy processes that we employ in the subsequent development. In Section 3 we develop the CGMY process as a time changed Brownian motion with drift, where the law of the time change is absolutely continuous over finite time intervals with respect to that of the one-sided stable $Y/2$ subordinator. The representation of CGMY as a time changed Brownian motion is described in Section 3. Section 4 presents the simulation details for CGMY as a time changed Brownian motion. Section 5 develops the time change for the Meixner process as absolutely continuous with respect to the one-sided stable $1/2$ subordinator. Simulation strategies for the Meixner process based on these representations are described in Section 6. Section 7 reports on the simulation results using prices of options computed by simulation and compared with those obtained by Fourier inversion. Section 8 provides conclusions.

2 PRELIMINARY RESULTS ON LÉVY PROCESSES

We present two results from the theory of Lévy processes that we make critical use of in our subsequent development. The first result relates the Lévy measure of a process obtained on subordinating a Brownian motion to the Lévy measure of the subordinator. The second result presents the detailed relationship between the standard presentation of the characteristic function of a two-sided jump and one-sided jump as a stable Lévy process and its Lévy measure. These are presented in the following two short sections.

2.1 Lévy measure of a subordinated Brownian motion

Suppose that the Lévy process $X(t)$ is obtained by subordinating Brownian motion with drift (ie, the process $\theta u + W(u)$, for $(W(u), u \geq 0)$ a Brownian motion) by an independent subordinator $Y(t)$ with Lévy measure $\nu(\mathrm{d}y)$. Then applying Theorem 30.1 of Sato (1999) (see also Huff (1969); Bertoin ((1999, p. 71); (1996, p. 162)) we get that the Lévy measure of the process $X(t)$ is given by $\mu(\mathrm{d}x)$ where:

$$\mu(\mathrm{d}x) = \mathrm{d}x \int_0^\infty \nu(\mathrm{d}y) \frac{1}{\sqrt{2\pi y}} \, \mathrm{e}^{-(x-\theta y)^2/2y} \tag{1}$$

2.2 Stable processes

The Stable Lévy process $\mathcal{S}(\sigma, \alpha, \beta) = (X(t), t \geq 0)$ with parameters (σ, α, β) (for details see DuMouchel (1973); DuMouchel (1975); Bertoin (1996); Samorodnitsky and Taqqu (1994); Nolan (2001) and Ito (2004)) has a characteristic function in standard form:

$$E[\mathrm{e}^{\mathrm{i}uX(t)}] = \exp(-t\Psi(u))$$

where the characteristic exponent $\Psi(u)$ is given by:

$$\begin{aligned} \Psi(u) &= \sigma^\alpha |u|^\alpha \left(1 - \mathrm{i}\beta \, \mathrm{sign}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right), \quad \alpha \neq 1 \\ &= \sigma|u|\left(1 + \mathrm{i}\beta \, \mathrm{sign}(u) \frac{2}{\pi} \log(|u|)\right), \quad \alpha = 1 \end{aligned} \tag{2}$$

The parameters satisfy the restrictions $\sigma > 0$, $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$. The one-sided jump stable processes result when $\beta = 1$ and there are only positive jumps or $\beta = -1$, in which case there are only negative jumps.

The Lévy density of the stable process is of the form:

$$k(x) = \frac{c_p}{x^{1+\alpha}} \mathbf{1}_{x>0} + \frac{c_n}{|x|^{1+\alpha}} \mathbf{1}_{x<0} \tag{3}$$

and we have that:

$$\beta = \frac{c_p - c_n}{c_p + c_n} \tag{4}$$

It remains to express σ in terms of the parameters of the Lévy measure. In the one-sided case with only positive jumps we have:

$$\sigma = \left[\frac{c_p \Gamma(\alpha/2)\Gamma(1-\alpha/2)}{2\Gamma(1+\alpha)}\right]^{1/\alpha} \tag{5}$$

and more generally for the two-sided jump case we have:

$$\sigma = \left[\frac{c_p + c_n}{2} \frac{\Gamma(\alpha/2)\Gamma(1-\alpha/2)}{\Gamma(1+\alpha)} \right]^{1/\alpha} \tag{6}$$

Conversely, c_p and c_n may be computed in terms of β and σ.

3 CGMY AS TIME CHANGED BROWNIAN MOTION

We wish to write the CGMY process in the form:

$$X_{\mathrm{CGMY}}(t) = \theta Y(t) + W(Y(t))$$

for an increasing time change process given by a subordinator $(Y(t), t \geq 0)$ independent of the Brownian motion $(W(s), s \geq 0)$.

The characteristic function of the CGMY process is:

$$E[\exp(\mathrm{i}uX_{\mathrm{CGMY}}(t))] = (\phi_{\mathrm{CGMY}}(u))^t = \exp\left(tC\Gamma(-Y) \begin{bmatrix} (M - \mathrm{i}u)^Y - M^Y + \\ (G + \mathrm{i}u)^Y - G^Y \end{bmatrix} \right)$$

The complex exponentiation is defined via the complex logarithm with a branch cut on the negative real axis with polar coordinate arguments for the complex logarithm restricted to the interval $]-\pi, +\pi]$. The CGMY process is defined as a pure jump Lévy process by its Lévy measure:

$$k_{\mathrm{CGMY}}(x) = C\left[\frac{\exp(-G|x|)}{|x|^{1+Y}} \mathbf{1}_{x<0} + \frac{\exp(-Mx)}{x^{1+Y}} \mathbf{1}_{x>0} \right]$$

On the other hand we have, in all generality, by conditioning on the time change that:

$$\begin{aligned} E[\mathrm{e}^{\mathrm{i}u(\theta Y(t) + W(Y(t)))}] &= E\left[\exp\left(\mathrm{i}u\theta Y(t) - \frac{Y(t)}{2} u^2 \right) \right] \\ &= E\left[\exp\left(-\left(\frac{u^2}{2} - \mathrm{i}u\theta \right) Y(t) \right) \right] \end{aligned}$$

Take $u(\lambda)$ to be any solution of:

$$\lambda = \left(\frac{u^2}{2} - \mathrm{i}u\theta \right)$$

Then we have the Laplace transform of the time change subordinator as:

$$E[\mathrm{e}^{-\lambda Y(t)}] = \exp(tC\Gamma(-Y)[(M - \mathrm{i}u(\lambda))^Y - M^Y + (G + \mathrm{i}u(\lambda))^Y - G^Y])$$

The solutions for u are:

$$u = \mathrm{i}\theta \pm \sqrt{2\lambda - \theta^2}$$

where we suppose that $\theta^2 < 2\lambda$.

We shall see that a good choice for θ, for sufficiently large λ permitting $\theta^2 < 2\lambda$, is:

$$\theta = \frac{G-M}{2}$$

and in this case:

$$M - \mathrm{i}u = \frac{G+M}{2} + \mathrm{i}\sqrt{2\lambda - \left(\frac{G-M}{2}\right)^2}$$
$$G + \mathrm{i}u = \frac{G+M}{2} - \mathrm{i}\sqrt{2\lambda - \left(\frac{G-M}{2}\right)^2}$$

It follows that the Laplace transform of the subordinator is:

$$E[\mathrm{e}^{-\lambda Y(t)}] = \exp(tC\Gamma(-Y)[2r^Y \cos(\eta Y) - M^Y - G^Y])$$
$$r = \sqrt{2\lambda + GM}$$
$$\eta = \arctan\left(\frac{\sqrt{2\lambda - ((G-M)/2)^2}}{((G+M)/2)}\right)$$

In the special case of $G = M$ we have:

$$E[\mathrm{e}^{-\lambda Y(t)}] = \exp\left(2tC\Gamma(-Y)\left[(2\lambda + M^2)^{Y/2}\cos\left(Y\arctan\left(\frac{\sqrt{2\lambda}}{M}\right)\right) - M^Y\right]\right)$$

3.1 The explicit time change for CGMY

We show that the time change subordinator $Y(t)$ associated with the CGMY process is absolutely continuous with respect to the one-sided stable $Y/2$ subordinator and, in particular, that its Lévy measure $\nu(\mathrm{d}y)$ takes the form:

$$\begin{aligned} \nu(\mathrm{d}y) &= \frac{K}{y^{1+Y/2}} f(y)\,\mathrm{d}y \\ f(y) &= \mathrm{e}^{-(B^2-A^2)y/2} E[\mathrm{e}^{-(B^2 y/2)(\gamma_{Y/2}/\gamma_{1/2})}] \\ B &= \frac{G+M}{2} \\ K &= \left[\frac{C\Gamma(Y/4)\Gamma(1-Y/4)}{2\Gamma(1+Y/2)}\right] \end{aligned} \tag{7}$$

where $\gamma_{Y/2}$, $\gamma_{1/2}$ are two independent gamma variates with unit scale parameters and shape parameters $Y/2$, $1/2$ respectively. Furthermore, we also observe that the Lévy measure of the CGMY subordinator may be written in terms of Hermite functions explicitly and we evaluate the expectation in (7) in terms of the Hermite functions as follows:

$$E[\mathrm{e}^{-(B^2 y/2)(\gamma_{Y/2}/\gamma_{1/2})}] = \frac{\Gamma(Y)}{\Gamma(Y/2)2^{Y/2-1}} h_{-Y}(B\sqrt{y})$$

where $h_\nu(z)$ is the Hermite function with parameter $\nu = -Y$ and is defined here by the integral representation:

$$h_\nu(z) = \frac{1}{\Gamma(-\nu)} \int_0^\infty \mathrm{e}^{-y^2/2-yz} y^{-\nu-1}\, \mathrm{d}y, \quad (\nu < 0)$$

(see, eg, Lebedev (1972, pp. 290–291) for an alternative representation).

3.2 Determining the time change for CGMY

For an explicit evaluation of the time change we begin by writing the CGMY Lévy density in the form:

$$k_{\mathrm{CGMY}}(x) = C\frac{\mathrm{e}^{Ax-B|x|}}{x^{1+Y}}, \quad \text{where: } A = \frac{G-M}{2}; \; B = \frac{G+M}{2}$$

Henceforth, when we encounter a Lévy measure $\mu(\mathrm{d}x)$ that is absolutely continuous with respect to Lebesgue measure we shall denote its density by $\mu(x)$. We now employ the result (1) and seek to find a Lévy measure $\nu_A(\mathrm{d}y)$ of a subordinator for the asymmetric case with asymmetry parameter A, satisfying:

$$\begin{aligned} C\frac{\mathrm{e}^{Ax-B|x|}}{|x|^{1+Y}} &= \int_0^\infty \nu_A(\mathrm{d}y)\frac{1}{\sqrt{2\pi y}}\, \mathrm{e}^{-(x-\theta y)^2/2y} \\ &= \int_0^\infty \nu_A(\mathrm{d}y)\frac{1}{\sqrt{2\pi y}}\, \mathrm{e}^{-(x^2/2y)-(\theta^2 y/2)+\theta x} \end{aligned}$$

We set $\theta = A$ and observe that the right choice for θ is $(G-M)/2$ as remarked earlier, and identify $\nu_A(\mathrm{d}y)$ such that:

$$C\frac{\mathrm{e}^{-B|x|}}{|x|^{1+Y}} = \int_0^\infty \nu_A(\mathrm{d}y)\frac{1}{\sqrt{2\pi y}}\, \mathrm{e}^{-(x^2/2y)-(\theta^2 y/2)} \tag{8}$$

Now taking:

$$C = \frac{\Gamma(Y/2)\Gamma(1-Y/2)}{\Gamma(1+Y/2)}$$

we recognize that the Lévy measure for the CGMY is that of the symmetric stable Y Lévy process with Lévy measure tilted as:

$$k_{\mathrm{CGMY}}(x) = \mathrm{e}^{Ax-B|x|} k_{\mathrm{Stable}(Y)}(x)$$

We also know that:

$$X_{\mathrm{Stable}(Y)}(t) = B_{Y^0(t)}$$

where $Y^0(t)$ is the one-sided stable $Y/2$ subordinator, independent of the Brownian motion (B_u).

We now write:

$$X_{\mathrm{CGMY}}(t) = \theta Y^{(1)}(t) + W_{Y^{(1)}(t)}$$

and we seek to relate the Lévy measures $\nu_A^{(1)}$ and $\nu^{(0)}$ of the processes $Y^{(1)}$ and $Y^{(0)}$.

From the result (1) we may write:

$$\mu_0(x) = \int_0^\infty \nu^{(0)}(\mathrm{d}y) \frac{\mathrm{e}^{-x^2/2y}}{\sqrt{2\pi y}}$$

$$\mu_1(x) = \int_0^\infty \nu_A^{(1)}(\mathrm{d}y) \frac{\mathrm{e}^{-(x-\theta y)^2/2y}}{\sqrt{2\pi y}}$$

Hence, we must have that:

$$\int_0^\infty \nu_A^{(1)}(\mathrm{d}y) \frac{\mathrm{e}^{-(x-\theta y)^2/2y}}{\sqrt{y}} = \mathrm{e}^{Ax-B|x|} \int_0^\infty \nu^{(0)}(\mathrm{d}y) \frac{\mathrm{e}^{-x^2/2y}}{\sqrt{y}}$$

Taking $\theta = A$, we obtain:

$$\int_0^\infty \nu_A^{(1)}(\mathrm{d}y) \frac{\mathrm{e}^{-(x^2/2y)-(A^2y/2)}}{\sqrt{y}} = \mathrm{e}^{-B|x|} \int_0^\infty \nu^{(0)}(\mathrm{d}y) \frac{\mathrm{e}^{-x^2/2y}}{\sqrt{y}}$$

Given the independence of the right-hand side from the asymmetry parameter A it is clear, using the notation $\nu^{(1)}(\mathrm{d}y)$ for the Lévy measure for the symmetric case with $A = 0$, that:

$$\nu_A^{(1)}(\mathrm{d}y) = \nu^{(1)}(\mathrm{d}y)\,\mathrm{e}^{A^2y/2}$$

and it remains to determine the solution to the symmetric case $\nu^{(1)}(\mathrm{d}y)$ satisfying:

$$\begin{aligned}\int_0^\infty \nu^{(1)}(\mathrm{d}y) \frac{\mathrm{e}^{-x^2/2y}}{\sqrt{y}} &= \mathrm{e}^{-B|x|} \int_0^\infty \nu^{(0)}(\mathrm{d}y) \frac{\mathrm{e}^{-x^2/2y}}{\sqrt{y}} \\ &= \frac{K\,\mathrm{e}^{-B|x|}}{|x|^{1+Y}} \end{aligned} \tag{9}$$

3.3 The time change in the symmetric case

We start with the symmetric Lévy measure in the form:

$$\mu(\mathrm{d}x) = C\frac{\mathrm{e}^{-B|x|}}{|x|^{1+Y}}, \quad \text{for } 0 < Y < 2 \tag{10}$$

PROPOSITION 1 *The measure (10) may be written as:*

$$\mu(\mathrm{d}x) = \left(\int_0^\infty \rho(\mathrm{d}a) \exp(-|x|a)\right) \mathrm{d}x$$

where:

$$\rho(\mathrm{d}a) = \mathrm{d}a \frac{C((a-B)^+)^Y}{\Gamma(Y+1)}$$

PROOF

$$\frac{\mu(\mathrm{d}x)}{\mathrm{d}x} = \frac{\mathrm{e}^{-B|x|}}{|x|^{1+Y}} = \mathrm{e}^{-B|x|} \int_0^\infty \frac{a^Y\,\mathrm{e}^{-|x|a}}{\Gamma(Y+1)}\,\mathrm{d}a$$
$$= \int_0^\infty \frac{a^Y\,\mathrm{e}^{-|x|(a+B)}}{\Gamma(Y+1)}\,\mathrm{d}a = \int_B^\infty \frac{(a-B)^Y\,\mathrm{e}^{-|x|a}}{\Gamma(Y+1)}\,\mathrm{d}a$$
$$= \int_0^\infty \frac{((a-B)^+)^Y\,\mathrm{e}^{-|x|a}}{\Gamma(Y+1)}\,\mathrm{d}a \qquad \square$$

We now develop three representations for the Lévy measure $\nu^{(1)}(\mathrm{d}y)$ of the Brownian subordinator in the symmetric case. The first representation is in terms of Hermite functions, the second representation is in terms of a ratio of independent gamma variates and third is an integral representation.

PROPOSITION 2 *The symmetric case subordinator has Lévy measure* $\nu^{(1)}(\mathrm{d}y)$ *given by:*

$$\nu^{(1)}(\mathrm{d}y) = \frac{1}{y^{1+Y/2}}\,\mathrm{e}^{-B^2y/2} h_{-Y}(B\sqrt{y})\,\mathrm{d}y \tag{11}$$

PROOF From Proposition 1 we may write:

$$\mu(\mathrm{d}x) = \mathrm{d}x \int_0^\infty \rho(\mathrm{d}a)\,\mathrm{e}^{-|x|a} \tag{12}$$

We now use the fact that:

$$\mathrm{e}^{-|x|a} = \int_0^\infty \frac{a}{\sqrt{2\pi u^3}}\,\mathrm{e}^{-(a^2/2u)-(x^2/2)u}\,\mathrm{d}u$$

substitute in the above integral, change variables to $y = 1/u$ and reverse the order of integration to obtain that:

$$\mu(\mathrm{d}x) = \mathrm{d}x \int_0^\infty \mathrm{d}y \frac{1}{\sqrt{2\pi y}}\,\mathrm{e}^{-x^2/2y} \int_0^\infty \rho(\mathrm{d}a) a\,\mathrm{e}^{-a^2y/2} \tag{13}$$

from which it follows on comparing with (9) that:

$$\nu^{(1)}(\mathrm{d}y) = \mathrm{d}y \int_0^\infty \rho(\mathrm{d}a) a\,\mathrm{e}^{-a^2y/2}$$
$$= \mathrm{d}y \frac{C}{\Gamma(Y+1)} \int_B^\infty \mathrm{d}a (a-B)^Y a\,\mathrm{e}^{-a^2y/2}$$

On integrating by parts we obtain:

$$\begin{aligned}\frac{\nu^{(1)}(\mathrm{d}y)}{\mathrm{d}y} &= \frac{C}{\Gamma(Y+1)y}\int_B^\infty \mathrm{e}^{-a^2y/2}Y(a-B)^{Y-1}\,\mathrm{d}a\\ &= \frac{C}{\Gamma(Y)y}\,\mathrm{e}^{-B^{2y}/2}\int_0^\infty \mathrm{d}b\,\mathrm{e}^{-(b^2y/2+bBy)}b^{Y-1}\\ &= \frac{C}{\Gamma(Y)}\frac{1}{y^{1+Y/2}}\,\mathrm{e}^{-B^2y/2}\int_0^\infty \mathrm{e}^{-(x^2/2+xB\sqrt{y})}x^{Y-1}\,\mathrm{d}x\\ &= \frac{1}{y^{1+Y/2}}\,\mathrm{e}^{-B^2y/2}h_{-Y}(B\sqrt{y})\end{aligned}$$ □

REMARK 3 Cherny and Shiryaev (2002, Theorem 3.17) discuss representations of the form (12) and (13) and show that these are both necessary and sufficient for a symmetric Lévy process to be a Brownian motion time changed by an independent subordinator.

REMARK 4 The representation (11) may also be related to the Whittaker parabolic cylinder function. This form of the subordinator for the CGMY has been noted by Cont and Tankov (2004, Remark 4.2, p. 120) and was also derived by Madan and Yor (2005).

We now develop another representation of the Lévy measure $\nu^{(1)}(\mathrm{d}y)$ of the subordinator in the symmetric case in terms of the Laplace transform of the ratio of $\gamma_{Y/2}$ to $\gamma_{1/2}$, two independent gamma variates. This representation is useful in developing simulations as we clearly see that the process $Y^{(1)}(t)$ for the required subordinator is absolutely continuous with respect to the one-sided stable $Y/2$ subordinator with a Radon–Nikodym derivative that is strictly below unity and we may appeal to Asmussen and Rosinski (2001) in this case and simulate by shaving the jumps of the one-sided stable $Y/2$ subordinator.

PROPOSITION 5 *The Lévy measure of the subordinator in the symmetric case can also be written as:*

$$\nu^{(1)}(\mathrm{d}y) = \mathrm{d}y\frac{2^{Y/2-1}\Gamma(Y/2)}{\Gamma(Y)}\frac{1}{y^{1+Y/2}}\,\mathrm{e}^{-B^2y/2}E\left[\exp\left(-\frac{B^2y}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}\right)\right] \tag{14}$$

for two independent gamma variates $\gamma_{Y/2}$, $\gamma_{1/2}$.

PROOF We first determine the constant term on considering the limit of:

$$\lim_{y\to 0}\left(y^{1+Y/2}\frac{\nu^{(1)}(\mathrm{d}y)}{\mathrm{d}y}\right) = h_{-Y}(0) = \frac{2^{Y/2-1}\Gamma(Y/2)}{\Gamma(Y)}$$

It then remains to show that:

$$E\left[\exp\left(-\frac{B^2y}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}\right)\right] = \theta h_{-Y}(B\sqrt{y})$$

for some constant θ that depends on Y.

For this we use the fact that $2\gamma_{1/2} \overset{(d)}{=} N^2$ the square of a standard normal variate and that $1/N^2 \overset{(d)}{=} T$ where T is a stable (1/2) with:

$$E[\mathrm{e}^{-\lambda T}] = \mathrm{e}^{-\sqrt{2\lambda}}$$

It follows that, with $x = B^2 y$:

$$\begin{aligned}
E[\mathrm{e}^{-(x/2)(\gamma_{Y/2}/\gamma_{1/2})}] &= E[\exp(-\sqrt{2x\gamma_{Y/2}})] \\
&= \frac{1}{\Gamma(Y/2)} \int_0^\infty \mathrm{d}\gamma\ \gamma^{Y/2-1}\ \mathrm{e}^{-\gamma}\ \mathrm{e}^{-\sqrt{2x\gamma}} \\
&= \frac{1}{\Gamma(Y/2)} \int_0^\infty \mathrm{d}u \left(\frac{u^2}{2}\right)^{Y/2-1} \mathrm{e}^{-(u^2/2)-u\sqrt{x}} \\
&= \frac{\Gamma(Y)}{\Gamma(Y/2)2^{Y/2-1}} h_{-Y}(\sqrt{x}) \\
&= \theta(Y) h_{-Y}(B\sqrt{y}) \qquad \square
\end{aligned}$$

Finally, with the help of beta–gamma algebra (see, eg, Chaumont and Yor (2003, exercise 4.6)) we may give yet another integral expression for $\nu^{(1)}(\mathrm{d}y)$. In fact we have the following.

PROPOSITION 6 *The subordinator* $\nu^{(1)}(\mathrm{d}y)$ *in the symmetric case may be written as:*

$$\nu^{(1)}(\mathrm{d}y) = \mathrm{d}y \frac{2^{Y/2-1}\Gamma(Y/2)}{\Gamma(Y)} \frac{B\ \mathrm{e}^{-B^2 y/2}}{y^{(Y+1)/2}} \int_0^\infty \frac{\mathrm{d}h}{\sqrt{2\pi}}\ \mathrm{e}^{-(B^2 y/2)h} \frac{h^{(Y-1)/2}}{(1+h)^{Y/2}} \tag{15}$$

PROOF Note on defining $Z = B^2 y/2$ that we need to establish that:

$$\frac{1}{\sqrt{Z}} E\left[\exp\left(-Z\left(\frac{\gamma_{Y/2}}{\gamma_{1/2}}\right)\right)\right] = K \int_0^\infty \mathrm{d}h\ \mathrm{e}^{-Zh} \frac{h^{(Y-1)/2}}{(1+h)^{Y/2}}$$

The constant K is identified by letting Z tend to zero on both sides and noting that on the right we have with $u = Zh$:

$$\frac{K}{\sqrt{Z}} \int_0^\infty \mathrm{d}u\ \mathrm{e}^{-u} u^{-1/2} = \frac{K\Gamma(1/2)}{\sqrt{Z}}$$

and hence $K = (1/\Gamma(1/2)) = (1/\sqrt{\pi})$.

We now write the reciprocal of $\sqrt{Z}$ on the left-hand side as an integral to obtain that the left-hand side is:

$$\begin{aligned}
&\frac{1}{\Gamma(1/2)} \int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}}\ \mathrm{e}^{-tZ} E\left[\exp\left(-Z\left(\frac{\gamma_{Y/2}}{\gamma_{1/2}}\right)\right)\right] \\
&= \int_0^\infty \frac{\mathrm{d}t}{\sqrt{\pi t}} E\left[\exp\left(-Z\left(t + \frac{\gamma_{Y/2}}{\gamma_{1/2}}\right)\right)\right]
\end{aligned}$$

Comparing this expression with the right-hand side and noting that we are taking Laplace transforms in Z of both sides, we deduce that for all test functions f we must have:

$$\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}} E\left[f\left(t + \frac{\gamma_{Y/2}}{\gamma_{1/2}}\right)\right] = \int_0^\infty \mathrm{d}h\ f(h) \frac{h^{(Y-1)/2}}{(1+h)^{Y/2}}$$

Defining $h = t + \gamma_{Y/2}/\gamma_{1/2}$ we observe that the expression on the left is:

$$\int_0^\infty \mathrm{d}h\ f(h) E\left[\frac{\mathbf{1}_{h > \gamma_{Y/2}/\gamma_{1/2}}}{\sqrt{h - \gamma_{Y/2}/\gamma_{1/2}}}\right]$$

and hence we need to show that:

$$E\left[\frac{\mathbf{1}_{h > \gamma_{Y/2}/\gamma_{1/2}}}{\sqrt{h - \gamma_{Y/2}/\gamma_{1/2}}}\right] = \frac{h^{(Y-1)/2}}{(1+h)^{Y/2}}$$

For this we note that, by beta–gamma algebra:

$$\frac{\gamma_{Y/2}}{\gamma_{1/2}} \stackrel{(d)}{=} \frac{1}{\beta_{1/2,Y/2}} - 1$$

and so for any test function:

$$\begin{aligned} E\left[g\left(\frac{\gamma_{Y/2}}{\gamma_{1/2}}\right)\right] &= \frac{1}{B(1/2, Y/2)} \int_0^1 g\left(\frac{1}{u} - 1\right) u^{-1/2}(1-u)^{Y/2-1}\,\mathrm{d}u \\ &= \frac{1}{B(1/2, Y/2)} \int_0^\infty g(v) \frac{v^{Y/2-1}}{(1+v)^{(Y+1)/2}}\,\mathrm{d}v \end{aligned}$$

In particular, the expectation of interest is:

$$\begin{aligned} &\frac{1}{B(1/2, Y/2)} \int_0^h \frac{1}{\sqrt{h-v}} \frac{v^{Y/2-1}}{(1+v)^{(Y+1)/2}}\,\mathrm{d}v \\ &\quad = \frac{h^{(Y-1)/2}}{B(1/2, Y/2)} \int_0^1 \frac{u^{Y/2-1}}{\sqrt{1-u}(1+uh)^{(Y+1)/2}}\,\mathrm{d}u \end{aligned}$$

and it remains to show that:

$$\frac{1}{(1+h)^{Y/2}} = \frac{1}{B(1/2, Y/2)} \int_0^1 \frac{u^{Y/2-1}}{\sqrt{1-u}(1+uh)^{(Y+1)/2}}\,\mathrm{d}u$$

This follows by noting that the left-hand side is the Laplace transform of $\gamma_{Y/2}$ while the right-hand side is the Laplace transform of the product of $\beta_{1/2,Y/2}$ and $\gamma_{(Y+1)/2}$, these variables being assumed independent. This is the product of a $\gamma_{Y/2}$ variate in distribution. □

REMARK 7 Note that on the right-hand side of (14) the power of y is $-(1+Y/2)$ while on the right-hand side of (15) it is $-((1+Y)/2)$.

REMARK 8 We briefly comment on the three representations for $\nu^1(\mathrm{d}y)$. Formula (11), which expresses $\nu^{(1)}(\mathrm{d}y)$ in terms of Hermite functions follows from the identification of the density of the symmetric CGMY Lévy measure as an integral of $\{\exp(-|x|a),\ a > 0\}$ and the well-known fact that:

$$\exp(-|x|a) = E\left[\exp\left(-\frac{x^2}{2}T_a\right)\right]$$

with T_a being the first hitting time of level a by Brownian motion.

The representation (14) hinges upon the expression of the Laplace transform of $(\gamma_{Y/2}/\gamma_{1/2})$ in terms of Hermite functions.

Formula (15) is derived via manipulations related to beta–gamma algebra, starting from formula (14).

We note that in our previous work (Madan and Yor (2005)) we arrived first at the representation (15) by considering that the Lévy density of the CGMY process is the product of $\exp(-B|x|)$ and the Lévy density of $(\beta_{\tau_t^{(Y/2)}},\ t \geq 0)$, where $(\tau_t^{(Y/2)},\ t \geq 0)$ is a stable $(Y/2)$ subordinator independent of the Brownian motion $(\beta_s,\ s \geq 0)$. Furthermore, in our previous paper (Madan and Yor (2005)) our representation of the Meixner process as subordinated Brownian motion is closer to the method used here for the representation (11) in that we identify directly the Lévy density of the Meixner process as a Laplace transform in x^2.

4 SIMULATING CGMY USING ROSINSKI REJECTION

We suppose that we have two Lévy measures $Q(\mathrm{d}x)$, $Q_0(\mathrm{d}x)$ with the property that:

$$\frac{\mathrm{d}Q}{\mathrm{d}Q_0} \leq 1$$

For the verification of the absolute continuity of the CGMY subordinators to the stable $Y/2$ subordinator we refer the reader to (Madan and Yor (2005, Sections 3.2.1 and 3.2.2)). In this case it is then shown by Asmussen and Rosinski (2001) that we may simulate the paths of Q from those of Q_0 by only accepting all jumps x in the paths of Q_0 for which:

$$\frac{\mathrm{d}Q}{\mathrm{d}Q_0}(x) > w$$

where w is an independent draw from a uniform distribution.

For our case we have that:

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0} = E[\mathrm{e}^{-yZ}] < 1$$

and so accept all jumps in the paths of ν_0 for which:

$$E[\mathrm{e}^{-yZ}] > w$$

4.1 Simulating CGMY as Brownian motion time changed by a shaved stable process

The focus here is on the simulation of the paths of the process. If we were to simulate a large step for the purpose of pricing a European option at some relatively large maturity, then one might use Fourier inversion to construct the distribution function at the large time step and employ standard inverse uniform methods to generate a CGMY random variable at the large time step. Alternatively, one may also simulate the CGMY process as a jump diffusion by simulating the large jumps using a compound Poisson process where jump sizes could be drawn from an incomplete gamma distribution function and the small jumps are approximated using a diffusion. For details on this simulation we refer the reader to Asmussen and Rosinski (2001). Yet another approach to valuation is to employ methods of measure changes as explained by Poirot and Tankov (2006).

For the simulation of the paths of the process, from the results of this paper we recognize that we have a time changed Brownian motion for which the paths may be constructed from the path of the time change. Since the time change is that of a shaved stable process, we follow classical methods described in Asmussen and Rosinski (2001) for the simulation of the stable process. Our contribution is the identification of the function to be used in shaving the stable process to construct the time change for Brownian motion appropriate for recovering the CGMY process.

The detailed algorithm is for parameters C, G, M, Y. First define the time step to be C:

$$t = C$$

Then we let:

$$A = \frac{G - M}{2}$$
$$B = \frac{G + M}{2}$$

We next simulate the path of a one-sided stable subordinator to time t using well-known methods. The Lévy measure for this subordinator is:

$$\frac{C\sqrt{2\pi}}{2^{Y/2}\Gamma((1+Y)/2)y^{Y/2+1}}\,\mathrm{d}y$$

The small jumps are truncated and replaced by their expected value at a drift rate of δ. The choice of the truncation point was determined with a view to controlling the absolute error in the distance between the true and approximate distribution function as described by Asmussen and Rosinski (2001, Theorem 3.1). We targeted an upper bound at 1%. The jumps above ε have an arrival rate of λ:

$$\delta = \frac{\varepsilon^{1-Y/2}}{1 - Y/2}; \qquad \lambda = \frac{2}{Y}\frac{1}{\varepsilon^{Y/2}}$$

The intervals between jump times are exponential and are simulated by:

$$t_i = -\frac{1}{\lambda}\log(1 - u_{2i})$$

for an independent uniform sequence u_{2i}. The actual jump times are:

$$\Gamma_j = \sum_{i=1}^{j} t_i$$

We generate the number of jumps required by a draw from an appropriate Poisson distribution. We then draw jump sizes for all these jumps up to the maximum number of jumps using an inverse uniform form as described by (16). We then recognize that jump times are uniform and sum over jumps up to a uniform time draw after the sizes have been deleted in accordance with (17) and (18).

For the jump magnitude we simulate from the normalized Lévy measure the jump size y_j given by:

$$y_j = \frac{\varepsilon}{(1 - u_{1j})^{2/Y}} \tag{16}$$

for an independent uniform sequence u_{1j}.

The process $S(t)$ for the stable subordinator is given by:

$$S(t) = \delta t + \sum_{j=1}^{\infty} y_j \mathbf{1}_{\Gamma_j < t}$$

We now obtain the CGMY subordinator $H(t)$ by:

$$H(t) = \delta t + \sum_{j=1}^{\infty} y_j \mathbf{1}_{\Gamma_j < t} \mathbf{1}_{h(y) > u_{3j}} \tag{17}$$

$$h(y) = \mathrm{e}^{-(B^2 - A^2)y/2} \frac{\Gamma(Y)}{\Gamma(Y/2)2^{Y/2-1}} h_{-Y}(B\sqrt{y}) \tag{18}$$

for an independent uniform sequence u_{3j}.

Finally we simulate the CGMY random variable by:

$$X = AH(t) + \sqrt{H(t)}z$$

for a draw z of a standard normal random variable.

The actual simulation cost in terms of random draws is of the order of that for the underlying one-sided stable process, as we merely throw away some of these jumps to build the CGMY subordinator and then make a draw from a normal distribution to obtain to the CGMY variate. The dependence of the cost on the parameters beyond that of simulating the stable process, is that of computing the truncation functions. On the selected settings the CPU time for one time step of a single day or $t = 0.004$ was 0.46 seconds for the CGMY process where we precomputed the truncation function and 3.31 seconds for the Meixner process studied below, where the truncation was not precomputed. The maximum number of jumps in the summation (17) for the settings used was 10 for CGMY. The corresponding value for Meixner was 11.

5 THE MEIXNER PROCESS AS A TIME CHANGED BROWNIAN MOTION

We consider the Meixner Process (Schoutens and Teugels (1998); Pitman and Yor (2003)) as a time changed Brownian motion. The Lévy measure of the Meixner process is:

$$k(x) = \delta \frac{\exp((b/a)x)}{x \sinh(\pi x/a)}$$

The characteristic function is given by:

$$\begin{aligned}\phi_{\text{Meixner}}(u) &= E[\mathrm{e}^{\mathrm{i}uX_1}] \\ &= \left(\frac{\cos(b/2)}{\cosh(au - \mathrm{i}b)/2)}\right)^{2\delta}\end{aligned}$$

To see this process as a time changed Brownian motion we wish to identify $l_A(u)$ the Lévy measure of a subordinator for the asymmetric case such that:

$$\begin{aligned}k(x) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{(x-Ay)^2}{2y}\right) l_A(y)\,\mathrm{d}y \\ &= \mathrm{e}^{Ax} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{x^2}{2y} - \frac{A^2 y}{2}\right) l_A(y)\,\mathrm{d}y\end{aligned}$$

Hence, we reduce to the symmetric case by setting:

$$A = \frac{b}{a}$$

and we then seek to write:

$$\delta \frac{1}{x \sinh(\pi x/a)} = \int_0^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{x^2}{2y} - \frac{A^2 y}{2}\right) l_A(y)\,\mathrm{d}y \tag{19}$$

We may introduce the Lévy measure for the symmetric case as $l(y)$ and we note that:

$$l_A(y) = l(y) \exp\left(\frac{A^2 y}{2}\right)$$

To identify the symmetric case we transform the left-hand side of (19) as follows.

We recall that:

$$\frac{Cx}{\sinh(Cx)} = E\left[\exp\left(-\frac{x^2}{2} T_C^{(3)}\right)\right]$$

where $T_C^{(3)} = \inf\{t \mid R_t^{(3)} = C\}$ for $R_t^{(3)}$ the $BES(3)$ process.

Then we write:

$$\begin{aligned}\delta \frac{1}{x \sinh(\pi x/a)} &= \frac{\delta(\pi x/a)}{(\pi x^2/a) \sinh(\pi x/a)} \\ &= \frac{\delta a}{\pi} \frac{1}{x^2} E\left[\exp\left(-\frac{x^2}{2} T_C^{(3)}\right)\right] \\ &= \frac{\delta a}{\pi} \frac{1}{x^2} E\left[\exp\left(-\frac{x^2 C^2}{2} T_1^{(3)}\right)\right]\end{aligned}$$

with $C = \pi/a$. Denote by $\theta(h)\,\mathrm{d}h$ the law of $T_1^{(3)}$. We may then write:

$$
\begin{aligned}
\delta \frac{1}{x\,\sinh(\pi x/a)} &= \frac{\delta a}{\pi} \int_0^\infty \frac{\mathrm{d}u}{2} \exp\left(-\frac{x^2 u}{2}\right) E\left[\exp\left(-\frac{x^2 C^2}{2} T_1^{(3)}\right)\right] \\
&= \frac{\delta a}{2\pi} \int_0^\infty \mathrm{d}u\; E\left[\exp\left(-\frac{x^2}{2}(u + C^2 T_1^{(3)})\right)\right] \\
&= \frac{\delta a}{2\pi} \int_0^\infty \mathrm{d}u \int_0^\infty \theta(t)\,\mathrm{d}t\, \exp\left(-\frac{x^2}{2}(u + C^2 t)\right) \\
&= \frac{\delta a}{2\pi} \int_0^\infty \mathrm{d}u \int_u^\infty \frac{\mathrm{d}v}{C^2} \exp\left(-\frac{x^2 v}{2}\right) \theta\left(\frac{v-u}{C^2}\right) \\
&= \frac{\delta a}{2\pi} \int_0^\infty \mathrm{d}v\, \exp\left(-\frac{x^2 v}{2}\right) \int_0^v \frac{\mathrm{d}u}{C^2} \theta\left(\frac{v-u}{C^2}\right) \\
&= \frac{\delta a}{2\pi} \int_0^\infty \mathrm{d}v\, \exp\left(-\frac{x^2 v}{2}\right) \int_0^{v/C^2} \mathrm{d}h\,\theta(h) \\
&= \int_0^\infty \mathrm{d}v\, \exp\left(-\frac{x^2 v}{2}\right) \widehat{\theta}(v)
\end{aligned}
$$

where:

$$
\begin{aligned}
\widehat{\theta}(v) &= \frac{\delta a}{2\pi} \int_0^{v/C^2} \theta(h)\,\mathrm{d}h \\
&= \frac{\delta a}{2\pi} P\left(T_1^{(3)} \le \frac{v}{C^2}\right) \\
&= \frac{\delta a}{2\pi} P\left(\max_{t \le v/C^2} R_t^{(3)} \ge 1\right)
\end{aligned}
$$

We recall that:

$$
T_1^{(3)} \overset{\text{(law)}}{=} \frac{1}{(\max_{t \le 1} R_t^{(3)})^2}
$$

We now transform the right-hand side of (19) to write:

$$
\begin{aligned}
&\int_0^\infty \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{x^2}{2y} - \frac{A^2 y}{2}\right) l(y)\,\mathrm{d}y \\
&\quad = \int_0^\infty \frac{1}{\sqrt{2\pi v^3}} \exp\left(-\frac{x^2 v}{2} - \frac{A^2}{2v}\right) l\left(\frac{1}{v}\right) \mathrm{d}v
\end{aligned}
$$

From the uniqueness of Laplace transforms we deduce that:

$$
\widehat{\theta}(v) = \frac{1}{\sqrt{2\pi v^3}} \exp\left(\frac{A^2}{2v}\right) l\left(\frac{1}{v}\right)
$$

or:

$$
\begin{aligned}
l(u) &= \sqrt{\frac{2\pi}{u^3}}\widehat{\theta}\left(\frac{1}{u}\right)\exp\left(-\frac{A^2u}{2}\right) \\
&= \sqrt{\frac{2\pi}{u^3}}\frac{\delta a}{2\pi}P(M_1^{(3)} \geq C\sqrt{u})\exp\left(-\frac{A^2u}{2}\right) \\
&= \frac{\delta a}{\sqrt{2\pi u^3}}P(M_1^{(3)} \geq C\sqrt{u})\exp\left(-\frac{A^2u}{2}\right) \\
&= \frac{\delta a}{\sqrt{2\pi u^3}}g(u)
\end{aligned}
$$

where, going back to the asymmetric case, we have:

$$
g(u) = P(M_1^{(3)} \geq C\sqrt{u})\exp\left(-\frac{A^2u}{2}\right)
$$

For the absolute continuity of our subordinator with respect to the one-sided stable $\frac{1}{2}$ we again refer the reader to (Madan and Yor (2005, Sections 3.2.1 and 3.2.2)).

For the simulation of Meixner as a time changed Brownian motion we would wish to evaluate:

$$
\begin{aligned}
P(M_1^{(3)} \geq C\sqrt{u}) &= P\left(\frac{1}{(M_1^{(3)})^2} \leq \frac{1}{C^2u}\right) \\
&= P\left(T_1^{(3)} \leq \frac{1}{C^2u}\right) \\
&= P\left(\pi^2 T_1^{(3)} \leq \frac{\pi^2}{C^2u}\right) \\
&= P\left(T_\pi^{(3)} \leq \frac{\pi^2}{C^2u}\right) \\
&= \sum_{-\infty}^{\infty}(-1)^n\,\mathrm{e}^{-n^2\pi^2/(2C^2u)}
\end{aligned}
$$

For the last equality we refer the reader to Pitman and Yor (2003).

6 SIMULATION OF THE MEIXNER PROCESS

The simulation strategy is similar to that employed in Section 4 for CGMY, except that here we first simulate the jumps of the one-sided stable $(1/2)$ with Lévy density:

$$
k(x) = \frac{\delta a}{\sqrt{2\pi x^3}}, \qquad x > 0
$$

and truncate using the function $g(y)$.

We approximate the small jumps of the subordinator using the drift ζ. The arrival rate is λ and the jump sizes are y_j obtained from an independent uniform sequence where:

$$\zeta = \delta a \sqrt{\frac{2\varepsilon}{\pi}}; \quad \lambda = \delta a \sqrt{\frac{2}{\pi \varepsilon}}; \quad y_j = \frac{\varepsilon}{u_j^2}$$

We then evaluate the function $g(y)$ at the point y_j and define the time-change variable:

$$\tau = \zeta + \sum_j y_j \mathbf{1}_{g(y_j) > w_j}$$

for yet another independent uniform sequence w_j.

The value of the Meixner random variable or, equivalently, the unit time level of the process is then generated as:

$$X = \frac{b}{a}\tau + \sqrt{\tau} z$$

where z is an independent standard normal variate.

7 RESULTS OF SIMULATIONS

In this section we present the results of simulating the processes at typical parameter values obtained on calibrating option prices on the S&P 500 index for both the CGMY and Meixner processes. The parameter values for the CGMY process were $C = 1$, $G = 5$, $M = 10$, and $Y = 0.5$. The parameters for the Meixner process were $a = 0.25$, $b = -0.5$ and $\delta = 1$.

Table 1 presents the option prices computed by simulation and by Fourier inversion using the methods of Carr and Madan (1999) for a range of strikes and maturities. The spot was 100 and we used an interest rate of 3% with a dividend yield of 1%. For strikes below the spot we employ puts while for strikes above or equal to the spot the prices are for call options.

8 CONCLUSION

Two Lévy processes, the CGMY process and the Meixner process are studied and it is shown that both processes can be represented as time changed Brownian motions. The time changes in both cases are absolutely continuous with respect to the one-sided stable α process with $\alpha = Y/2$ for CGMY and $1/2$ for Meixner. It is then possible to simulate both processes as shaved stable processes where one throws away some jumps from the one-sided stable α simulation. The simulation cost is that of the stable α process as the jumps to throw away are analytically determined and the other parameters enter only through this analytical truncation. An exercise on pricing European options by simulation has also been presented.

TABLE 1 Option prices via simulation and FFT.

Strike	Maturity	CGMY simulation	CGMY FFT	Meixner simulation	Meixner FFT
80	0.25	0.8780	0.8698	0.1174	0.1266
90	0.25	2.2482	2.2475	0.6022	0.6179
100	0.25	5.9277	5.8919	3.3690	3.4127
110	0.25	2.1573	2.1420	0.5934	0.6543
120	0.25	0.8100	0.7848	0.1293	0.1587
80	0.5	1.8725	1.8854	0.3150	0.3284
90	0.5	4.0581	4.0589	1.2362	1.2868
100	0.5	8.8413	8.8226	5.1135	5.2081
110	0.5	4.7237	4.7026	1.5889	1.6870
120	0.5	2.3855	2.3585	0.4656	0.5282
80	0.75	2.8716	2.8638	0.5593	0.5714
90	0.75	5.5181	5.5219	1.8617	1.9000
100	0.75	11.0472	11.0672	6.5256	6.6170
110	0.75	6.9062	6.8875	2.7066	2.7717
120	0.75	4.1357	4.1038	1.0410	1.0710
80	1	3.7272	3.7681	0.8117	0.8323
90	1	6.7255	6.7515	2.3644	2.4458
100	1	12.9427	12.9545	7.7168	7.8220
110	1	8.7682	8.7819	3.7458	3.8095
120	1	5.7451	5.7803	1.6819	1.7152

REFERENCES

Asmussen, S., and Rosinski, J. (2001). Approximations of small jumps of Lévy processes with a view towards simulation. *Journal of Applied Probability* **38**, 482–493.

Barndorff-Nielsen, O. E. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics* **2**, 41–68.

Bertoin, J. (1996). *Lévy Processes*. Cambridge University Press, Cambridge.

Bertoin, J. (1999). *Subordinators: Examples and Applications (St. Flour Summer School 1997)* (*Lecture Notes in Mathematics, Vol. 1717*). Springer, Berlin.

Boyarchenko, S. I., and Levendorskii, S. (1999). Generalizations of the Black–Scholes equation for truncated Lévy processes. Working Paper, University of Texas at Austin, Texas.

Boyarchenko, S. I., and Levendorskii, S. (2000). Option pricing for Truncated Lévy processes. *International Journal of Theory and Applications in Finance* **3**, 549–552.

Carr, P., and Madan, D. (1999). Option valuation using the fast Fourier transform. *The Journal of Computational Finance* **2**, 61–73.

Carr, P., Geman, H., Madan, D., and Yor, M. (2002). The fine structure of asset returns: an empirical investigation. *Journal of Business* **75**, 305–332.

Carr, P., Geman, H., Madan, D., and Yor, M. (2004). From local volatility to local Lévy models. *Quantitative Finance* **5**, 581–588.

Chaumont, L., and Yor, M. (2003). *Exercises in Probability: A Guided Tour from Measure Theory to Random Processes via Conditioning*. Cambridge University Press, Cambridge.

Chen, Q. (2008). Dependence structure in Lévy processes and its applications in finance. PhD Thesis, University of Maryland.

Cherny, A., and Shiryaev, A. (2002). Change of time and measure for Lévy processes. *Lectures for the Summer School: From Lévy Processes to Semimartingales-Recent Theoretical Developments and Applications to Finance*. Working Paper, University of Aarhus.

Cont, R., and Tankov, P. (2004). *Financial Modeling with Jump Processes* (*Financial Mathematics Series*). Chapman and Hall/CRC, London.

DuMouchel, W. H. (1973). Stable distribution in statistical inference 1: symmetric stable distributions compared to other symmetric long-tailed distributions. *Journal of the American Statistical Association* **68**, 469–477.

DuMouchel, W. H. (1975). Stable distributions in statistical inference 2: information from stably distributed samples. *Journal of the American Statistical Association* **70**, 386–393.

Eberlein, E., Keller, U., and Prause, K. (1998). New insights into smile, mispricing and value at risk. *Journal of Business* **71**, 371–406.

Gradshteyn, I. S., and Ryzhik, I. M. (1995). *Table of Integrals, Series and Products*. Academic Press, New York.

Grigelionis, B. (1999). Processes of Meixner Type. *Lithuanian Mathematics Journal* **39**, 33–41.

Huff, B. (1969). The strict subordination of a differential process. *Sankhya Series A* **31**, 403–412.

Ito, K. (2004). *Stochastic Processes*. Springer, Berlin.

Joshi, M., and Stacey, A. (2005). Intensity gamma: a new approach to pricing portfolio credit derivatives. Working Paper, Royal Bank of Scotland.

Koponen, I. (1995). Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process. *Physical Review* E **52**, 1197–1199.

Lebedev, N. N. (1972). *Special Functions and Their Applications*. Dover, New York.

Luciano, E., and Schoutens, W. (2006). A multivariate jump driven financial asset model. *Quantitative Finance* **6**, 385–402.

Madan, D., and Yor, M. (2005). CGMY and Meixner subordinators are absolutely continuous with respect to one-sided stable subordinators. Prepublication No 1044, Laboratoire de Probabilités, Modèles aléatoires, Paris VI.

Madan, D., Carr, P., and Chang, E. (1998). The variance gamma process and option pricing. *European Finance Review* **2**, 79–105.

Nolan. J. P. (2001). Maximum likelihood estimation and diagnostics for stable distributions. *Lévy processes—Theory and Application*, Barndorff-Nielsen, O. E., Mikosch, T. and Resnick, S. I. (eds). Birkhäuser, Boston, MA, pp. 379–400.

Pitman, J., and Yor, M. (2003). Infinitely divisible laws associated with hyperbolic functions. *Canadian Journal of Mathematics* **55**(2), 292–330.

Poirot, J., and Tankov, P. (2006). Monte Carlo option pricing for tempered stable (CGMY) processes. *Asia Pacific Financial Markets* **13**, 327–344.

Rosínski, J. (2001). Series representations of Lévy processes from the perspective of point processes. *Lévy processes—Theory and Application*. Barndorff-Nielsen, O. E., Mikosch, T. and Resnick, S. I. (eds). Birkhäuser, Boston, MA, pp. 401–415.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.

Samorodnitsky, G., and Taqqu, M. S. (1994). *Stable Non-Gaussian Random Processes*. Chapman and Hall, New York.

Schoutens, W. (2000). *Stochastic Processes and Orthogonal Polynomials* (*Lecture Notes in Statistics, Vol. 146*). Springer, Berlin.

Schoutens, W., and Teugels, J. L. (1998). Lévy processes, polynomials and martingales. *Communications in Statistics: Stochastic Models* **14**, 335–349.